Roll No. ....

D–3754

M. A./M. Sc. (Previous) EXAMINATION, 2020

MATHEMATICS

Paper Fourth

(Complex Analysis)

Time : Three Hours] [*Maximum Marks : 100*

Note : All questions are compulsory. Attempt any *two* Parts from each Unit. All questions carry equal marks.

Unit—I

1. (a) State and prove Cauchy-Goursat theorem.

(b) Let $f(z)$ be analytic within and on a closed contour C, and let a be any point within C. Then :

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

(c) If $f(z)$ is analytic within and on a closed contour C except at a finite number of poles and has no zeros on C. then :

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \mathrm{N} - \mathrm{P}$$

where N is the number of zeros and P is the number of poles inside C.

Unit—II

2. (a) Apply the calculus of residue to prove that :

$$\int_0^{\infty} \frac{\cos mx}{a^2 + x^2}\, dx = \frac{\pi}{2a} e^{-ma}$$

where $m \geq 0$, $a > 0$.

(b) State and prove Montel's theorem.

(c) Find the bilinear transformation which maps the points

$$z_1 = 2, z_2 = i, z_3 = -2$$

into the points

$$w_1 = 1, w_2 = i, w_3 = -1.$$

Unit—III

3. (a) State and prove Runge's theorem.

(b) Let (*f*, D) be a function element and let G be a region containing D such that (*f*, D) admits unrestricted continuation in G. Let $a \in \mathrm{D}, b \in \mathrm{G}$ and let γ_0 and γ_1 be paths in G from a to *b*. Let $\{(f_t, \mathrm{D}_t) : 0 \leq t \leq 1\}$ and $\{(g_t, \mathrm{D}_t) : 0 \leq t \leq 1\}$ be analytic continuations of (*f*, D) along γ_0 and γ_1 respectively. If γ_0 and γ_1 are fixed-end-point homotopic in G, then :

$$[f_1]_b = [g_1]_b$$

(c) State and prove Harnack's Inequality.

Unit—IV

4. (a) Let $f(z)$ be analytic in the closed disc $|z| \leq \mathrm{R}$. Assume that $f(0) \neq 0$ and no zeros of $f(z)$ lies on

$|z| = \mathrm{R}$. If $z_1, z_2, \ldots\ldots, z_n$ are the zeros of $f(z)$ in the open disc $|z| < \mathrm{R}$, each repeated as often as its multiplicity and $z = re^{i\theta}, 0 \leq r < \mathrm{R}, f(z) \neq 0$, then :

$$\log |f(z)| = -\sum_{i=1}^{n} \log \left| \frac{\mathrm{R}^2 - \bar{z}_i z}{\mathrm{R}(z - z_i)} \right|$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} \frac{(\mathrm{R}^2 - r^2) \log |f(\mathrm{R}e^{i\phi})|}{\mathrm{R}^2 - 2\mathrm{R}r\cos(\theta - \phi) + r^2}\, d\phi$$

(b) Let *f* be a non-constant function. Define :

$$\rho_1 = \inf \{\lambda \geq 0 : \mathrm{M}(r) \leq \exp(r^{\lambda})$$

for sufficiently larger}

$$\rho_2 = \limsup_{r \to \infty} \frac{\log \log \mathrm{M}(r)}{\log r}$$

then $\rho_1 = \rho_2$.

(c) If $f(z)$ is an entire function of order ρ and convergence exponent σ, then $\sigma \leq \rho$.

Unit—V

5. (a) Let f be analytic in $\mathrm{D} = \{z : |z| < 1\}$ and let $f(0) = 0, f'(0) = 1$ and $|f(z)| \leq \mathrm{M}$ for all z in D. Then $\mathrm{M} \geq 1$ and $f(\mathrm{D}) \supset \mathrm{B}\left(0; \frac{1}{6\mathrm{M}}\right)$.

(b) State and prove Montel Caratheodory theorem.

(c) State and prove the Great-Picard theorem.